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ANALYTIC SOLUTIONS TO THE CONSTRAINT EQUATION FOR A FORCE-FREE MAGNETOSPHERE AROUND A KERR BLACK HOLE

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ABSTRACT

The Blandford-Znajek constraint equation for a stationary, axisymmetric black-hole force-free magnetosphere is cast in a 3+1 absolute space and time formulation, following Komissarov (2004). We derive an analytic solution for fields and currents to the constraint equation in the far-field limit that satisfies the Znajek condition at the event horizon. This solution generalizes the Blandford-Znajek monopole solution for a slowly rotating black hole to black holes with arbitrary angular momentum. Energy and angular momentum extraction through this solution occurs mostly along the equatorial plane. We also present a nonphysical, reverse jet-like solution.

Subject headings: Black Hole Physics: Force Free Magnetospheres, Energy Extraction

1. INTRODUCTION

Penrose (1969) recognized the possibility to extract the spin energy of a black hole using particle decay in negative energy orbits within the ergosphere. Based on studies of force-free pulsar magnetospheres, Blandford & Znajek (1977) proposed that rotational energy could be extracted through currents flowing in the black hole's magnetosphere. In this picture, strong electric and magnetic fields are induced by gravito-MHD (GMHD) processes. Blandford & Znajek (1977) derived the equations for a stationary, axisymmetric force-free magnetosphere in curved spacetime, and reduced the set of equations to a central constraint equation relating toroidal magnetic field H_φ to the charge density ρ and toroidal current density J_φ. They also found a perturbative solution to the constraint equation valid in the limit $a/M \ll 1$, where M is the black hole mass and a is the angular momentum per unit mass.

Thorne & MacDonald (1982) and MacDonald & Thorne (1982) developed this theory in a more intuitive "3+1" formulation that led to the membrane paradigm (Thorne et al. 1986), where the equations of GMHD were written using the familiar electric and magnetic 3-vectors in absolute space whose time dependence is governed by Maxwell-type equations. Komissarov (2004) recently presented the essential equations of this formulation in a form useful for numerical studies, and helped resolve questions (Punsly & Coroniti 1990; Punsly 2001) relating to energy extraction in the membrane paradigm.

The equations presented by Komissarov (2004) provide a useful starting point to search for analytic solutions. Here we use these equations to rederive the constraint equation of Blandford & Znajek (1977) in the 3+1 form. This brings forth a clear understanding of the nature of the poloidal functions defining the currents and fields. We have discovered an analytical solution valid for arbitrary angular momentum that reduces to the monopole solution of Blandford & Znajek (1977) in the limit of $a/M \ll 1$. This solution, which satisfies the Znajek (1977) regularity condition, permits energy extraction preferentially along the equatorial direction of the Kerr black hole.

The modified Maxwell's equations in curved space-time are given in Section 2, and the equations for a force-free magnetosphere are presented in Section 3. In Section 4, we construct the form of fields and currents for a given poloidal function Ω. The governing constraint equation for this function is given in Section 5. Solutions to this equation are derived in Section 6, and we summarize in Section 7.

2. ELECTRODYNAMICS IN ABSOLUTE SPACE

While Maxwell's equations preserve all of its elegance in a covariant formalism on a four dimensional manifold, it distracts from some of the simple (far-field) solutions that it might permit. With this is mind, we briefly state the essential equations of electrodynamics in an absolute three dimensional space. The recent paper by Komissarov (2004) explains how these equations are derived.

The construction of absolute space is facilitated by noting that an arbitrary spacetime metric can be written in the form

$$ds^2 = (\beta^2 - \alpha^2)dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j. \quad (1)$$

The functions x^i serve as coordinates for our spacelike hypersurfaces defined by constant values of t. Consider one such hyperspace Σ defined by the region $t = 0$. We can think of electric and magnetic fields (E and B) as objects existing in our absolute space Σ. The time evolution equations for E and B in the presence of a charge density ρ and electric current density vector J in our absolute (curved) space endowed with a metric γ_{ij} are given by the following set of Maxwell's equations:

$$\nabla \cdot B = 0, \quad (2)$$

$$\partial_t B + \nabla \times E = 0, \quad (3)$$

and their inhomogeneous counterparts,

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$$\nabla \cdot D = \rho, \tag{4}$$

$$-\partial_t D + \nabla \times H = J. \tag{5}$$

It is important to remember that E, B, D, H, and J are vectors in our three dimensional absolute space Σ, and in general are time dependent. Also, ∇ is the covariant derivative induced by the metric γ_{ij} on Σ. As usual, the curl of a vector field is defined by the expression

$$(\nabla \times A)^i = \epsilon^{ijk}\nabla_j A_k, \tag{6}$$

where ϵ^{ijk} is the completely antisymmetric pseudotensor such that $\epsilon^{123} = \frac{1}{\sqrt{\gamma}}$, and $\gamma = \mathrm{Det}(\gamma_{ij})$. It is easily seen that Maxwell's equations imply the continuity equation

$$\partial_t \rho + \nabla \cdot J = 0. \tag{7}$$

Unlike its flat space counterparts, even in regions of negligible electric and magnetic susceptibilities, $E \neq D$, and $B \neq H$. Indeed, it can be shown that they instead satisfy the consitutive relations

$$E = \alpha D + \beta \times B, \tag{8}$$

and

$$H = \alpha B - \beta \times D. \tag{9}$$

Of interest are spacetimes admitting Killing fields corresponding to axial symmetry (m) and stationarity. Consequently, Noether's theorem imply energy and angular momentum conservation laws. They can be stated in the form

$$\partial_t e + \nabla \cdot S = -(E \cdot J), \tag{10}$$

and

$$\partial_t l + \nabla \cdot L = -(\rho E + J \times B) \cdot m. \tag{11}$$

Here,

$$e = \frac{1}{2}(E \cdot D + B \cdot H) \tag{12}$$

is the volume density of energy, and

$$l = (D \times B) \cdot m \tag{13}$$

is the density of angular momentum,

$$S = E \times H \tag{14}$$

is the flux of energy, and

$$L = -(E \cdot m)D - (H \cdot m)B + \frac{1}{2}(E \cdot D + B \cdot H)m \tag{15}$$

is the flux of angular momentum.

3. STATIONARY, AXISYMMETRIC FORCE FREE MAGNETOSPHERES

The condition that the magnetosphere is force free brings about enough structure into Maxwell's equations to enable the introduction of a streaming function that will help us visualize the field structure in geometric terms. It is traditional to use spheroidal spatial coordinates given by $x^i = (r, \theta, \varphi)$ such that $m = \partial_\varphi$. Assumptions of stationarity and axissymmetry imply that $\partial_\varphi g_{\mu\nu} = 0 = \partial_t g_{\mu\nu}$.

In our absolute space framework, the force free condition reduces to

$$E \cdot J = 0, \tag{16}$$

and

$$\rho E + J \times B = 0. \tag{17}$$

These restrictions, along with Maxwell's equations and Eqs. (8) and (9), imply that

$$E_T = 0, \tag{18}$$

and

$$E_P \cdot B_P = 0. \tag{19}$$

The poloidal and toroidal components (A_P, and A_T) of a vector field are defined such that $A = A_P + A_T$, where $A_P = A^r\partial_r + A^\theta\partial_\theta$ and $A_T = A^\varphi\partial_\varphi$. Eqs. (18) and (19) imply that there exists a vector $\omega = \Omega\partial_\varphi$ such that

$$E = -\,\omega \times B, \tag{20}$$

From the vanishing of the curl of E under the stationarity condition (Eq. (3)), one finds that

$$B \cdot \nabla\Omega = 0. \tag{21}$$

It can also be shown that

$$B \cdot \nabla H_\varphi = 0. \tag{22}$$

4. EXPLICIT EXPRESSIONS FOR FIELDS AND CURRENTS

To simplify calculations, we shall assume that the spatial coordinates are orthogonal, and that the shift vector β is purely toroidal, i.e., $\beta = (0, 0, \beta_\varphi)$. The Kerr solution written in Boyer-Lindquist (though not Kerr-Schild) coordinates can be written in this form.

Surfaces of constant Ω are referred to as poloidal surfaces (not to be confused with poloidal components of a vector). From Eq. (21) it is clear that B is tangent to poloidal surfaces. Since Ω does not have any φ dependence, and since Eq. (21) has nothing to say about the toroidal component of B, it is clear that B_P will entertain solutions of the type

$$B_P = \frac{\Lambda}{\sqrt{\gamma}}(-\Omega_{,\theta}\partial_r + \Omega_{,r}\partial_\theta) \tag{23}$$

where, for the moment, Λ is an arbitrary function. This must be so because in the two dimensional subspace given by $\Omega = const$, there is a unique vector (modulo magnitude) that is perpendicular to $\nabla\Omega$. The condition that B is divergence free means that Λ satisfies

$$\Lambda_{,r}\,\Omega_{,\theta} \;=\; \Lambda_{,\theta}\,\Omega_{,r}. \tag{24}$$

Consequently, Λ is a poloidal function (a function that is constant on poloidal surfaces). In the notation of the original paper by Blandford & Znajek (1977), $\Lambda d\Omega \equiv -dA_\varphi$. The electric field is immediately calculated from Eq. (20) and, as expected, is the gradient of a scalar function:

$$E_P = \Lambda d(\Omega^2/2) = d\int \Lambda\Omega d\Omega\,. \tag{25}$$

From Eq. (8), we see that

$$D = D_P = \frac{\Lambda}{\alpha}(\Omega + \beta^\varphi)d\Omega. \tag{26}$$

Similarly, the expression for H_P can be calculated from Eq. (9), giving

$$H_P = (\alpha^2 - \beta^2 - \beta_\varphi \Omega)\frac{B_P}{\alpha}. \tag{27}$$

The electric charge is determined by the divergence of the D_P, (Eq. (4)). Explicitly,

$$\sqrt{\gamma}\rho = \partial_r[\frac{\Lambda}{\alpha\sqrt{\gamma}}(\gamma_{\varphi\varphi}\Omega + \beta_\varphi)\gamma_{\theta\theta}\Omega_{,r}] + \partial_\theta[\frac{\Lambda}{\alpha\sqrt{\gamma}}(\gamma_{\varphi\varphi}\Omega + \beta_\varphi)\gamma_{rr}\Omega_{,\theta}]. \tag{28}$$

The toroidal component of the electric current density vector can be obtained from the derivatives of components of H_P:

$$\sqrt{\gamma}J^\varphi = H_{\theta,r} - H_{r,\theta} = \partial_r[\frac{\Lambda}{\alpha\sqrt{\gamma}}(\alpha^2 - \beta^2 - \beta_\varphi\Omega)\gamma_{\theta\theta}\Omega_{,r}] + \partial_\theta[\frac{\Lambda}{\alpha\sqrt{\gamma}}(\alpha^2 - \beta^2 - \beta_\varphi\Omega)\gamma_{rr}\Omega_{,\theta}]. \tag{29}$$

It clear from the above discussion that the poloidal fields and, consequently, the toroidal current J^φ are uniquely described by the poloidal functions Ω and Λ. On the other hand, the toroidal fields and the poloidal currents can be determined from the poloidal function H_φ. In particular, from Eq. (9), it is clear that $H_\varphi = \alpha B_\varphi$. Maxwell's equation (Eq. (5)) implies that

$$\sqrt{\gamma}J_p = H_{\varphi,\theta}\partial_r - H_{\varphi,r}\partial_\theta. \tag{30}$$

Thus we see that fields and currents separate into two distinct categories: objects that are determined by Ω and Λ, and those that are determined by H_φ. Outside of the fact that H_φ is a poloidal function (by definition, Ω is), it is not yet clear as to how these two functions are dynamically related. This issue will be cleared up in the following section.

5. THE CONSTRAINT EQUATION

The expressions for the fields and currents given in the previous section naturally satisfies Eq. (16). Since the toroidal component of the electric field vanishes, it is easily checked that, from Eq. (17), $(J \times B)_\varphi = 0$ (as shown below in Eq. (32)). Thus the only remaining requirements for a force-free solution is

$$\rho E_P + (J \times B)_P = 0. \tag{31}$$

The implication of the above equation is most easily understood by projecting the equation onto E_P, B_p, which serve as a basis vectors for poloidal vector fields. The above equation yields no constraint when projected onto B_P, i.e.,

$$\rho E \cdot B_p + (J \times B) \cdot B_P = (J \times B) \cdot (B - B_T) = - B^\varphi (J \times B)_\varphi = -B^\varphi \Lambda \frac{1}{\sqrt{\gamma}}(H_{\varphi,\theta}\Omega_{,r} - H_{\varphi,r}\Omega_{,\theta}) = 0. \tag{32}$$

Projecting Eq. (17) onto E_P gives

$$\rho E \cdot E_p + (J \times B) \cdot E_P = \rho E^2 + ((J_P + J_T) \times (B_P + B_T)) \cdot E_P = \rho E^2 + ((J_P \times B_T) + (J_T \times B_P)) \cdot E_P = 0, \tag{33}$$

since J_P is parallel to B_P. With the help of the following relations,

$$J_P = \frac{1}{\sqrt{\gamma}}\frac{dH_\varphi}{d\Omega}(\Omega_{,\theta}\partial_r - \Omega_{,r}\partial_\theta) = -\frac{dH_\varphi}{\Lambda d\Omega}B_P,$$

$$E_r B_\theta - E_\theta B_r = \frac{\Omega\Lambda^2}{\sqrt{\gamma}}(\gamma_{\theta\theta}(\Omega_{,r})^2 + \gamma_{rr}(\Omega_{,\theta})^2),$$

and

$$E^2 = \frac{\Omega^2\Lambda^2}{\gamma_{rr}\gamma_{\theta\theta}}(\gamma_{\theta\theta}(\Omega_{,r})^2 + \gamma_{rr}(\Omega_{,\theta})^2), \tag{34}$$

Eq. (33) reduces to the manageable form

$$\frac{1}{2\Lambda}\frac{dH_\varphi^2}{d\Omega} = \alpha(\rho\Omega\gamma_{\varphi\varphi} - J_\varphi). \tag{35}$$

This is the final and only constraint equation. If Ω and Λ are picked such that the right hand side of the above equation is a poloidal function, then H_φ continues to be poloidal function. The poloidal functions Ω, Λ, and H_φ then uniquely determines all currents and fields. It is important to realize that Ω is not to be thought of as a potential: physically relevant quantities like the electric field depend on Ω directly, and are not invariant transformations of the type $\Omega \to \Omega + const$. The charge density ρ and the toroidal current J_φ are functions of Ω and Λ (see Eqs. (28) and (29)).

6. ASYMPTOTIC SOLUTIONS AND ENERGY EXTRACTION

Inserting Eqs. (28) and (29) into Eq. (35), our constraint equation gives

$$\frac{1}{2\Lambda}\frac{dH_\varphi^2}{d\Omega} = \frac{\alpha\gamma_{\varphi\varphi}}{\sqrt{\gamma}}[\Omega\partial_r(\frac{\Lambda}{\alpha\sqrt{\gamma}}(\gamma_{\varphi\varphi}\Omega + \beta_\varphi)\gamma_{\theta\theta}\Omega_{,r}) + \Omega\partial_\theta(\frac{\Lambda}{\alpha\sqrt{\gamma}}(\gamma_{\varphi\varphi}\Omega + \beta_\varphi)\gamma_{rr}\Omega_{,\theta}) + \partial_r(\frac{\Lambda}{\alpha\sqrt{\gamma}}(\beta^2 - \alpha^2 + \beta_\varphi\Omega)\gamma_{\theta\theta}\Omega_{,r}) + \partial_\theta(\frac{\Lambda}{\alpha\sqrt{\gamma}}(\beta^2 - \alpha^2 + \beta_\varphi\Omega)\gamma_{rr}\Omega_{,\theta})]. \tag{36}$$

Therefore, Eq. (35) is equivalent to Eq. (3.14) of Blandford & Znajek (1977) written in the 3+1 formalism.

While searching for solutions for the fields and currents that might permit extraction of energy and angular momentum from a rotating black hole, it is advantageous to observe that

$$\frac{d^2\mathcal{E}}{dAdt} = S^r\sqrt{\gamma_{rr}} = -H_\varphi\Omega B^r\sqrt{\gamma_{rr}}, \tag{37}$$

$$\frac{d^2\mathcal{L}}{dAdt} = L^r\sqrt{\gamma_{rr}} = -H_\varphi B^r\sqrt{\gamma_{rr}}, \tag{38}$$

as can be see from Eqs. (14) and (15). Here $\mathcal{E}$ and $\mathcal{L}$ are the total energy and angular momentum, respectively, extracted from the black hole.

For definiteness, we shall consider the magnetosphere of a Kerr black hole in Boyer-Lindquist coordinates. For finite rates of energy and angular momentum extraction, it is clear from the above two equations that for $r \gg M$, $\Omega \rightarrow \Omega(\theta)$. With this in mind, we seek solutions to the constraint equation of the type $\Omega = \Omega(\theta)$ for all values of r. This means that all poloidal functions are functions of θ alone, since all poloidal functions are of "zeroeth" order in r. Due to the inherent complexity of the constraint equation, we shall further consider Eq. (36) in the far field limit. To order $(1/r^3)$ for strictly θ-dependent functions, Eq. (36) takes the form:

$$\begin{aligned}
-\frac{1}{2f(\theta)}\frac{dH_\varphi^2}{d\theta} &= -\Omega\sin\theta\frac{d}{d\theta}(f\Omega\sin\theta) \\
&+\frac{\sin\theta}{r^2}[-a^2\Omega\sin^2\theta\frac{d}{d\theta}(f\Omega\sin\theta)+\frac{d}{d\theta}(\frac{f}{\sin\theta})] \\
&+2M\frac{\sin\theta}{r^3}[a\Omega\frac{d}{d\theta}(f\sin\theta(1-a\Omega\sin^2\theta))- \\
&\frac{d}{d\theta}(\frac{f}{\sin\theta}(1-a\Omega\sin^2\theta))],
\end{aligned} \tag{39}$$

where M and a are the mass and the angular momentum per unit mass of the black hole, respectively, and $f(\theta) \equiv -\Lambda\Omega_{,\theta} \equiv A_{\varphi,\theta}$. For a consistent formulation of the theory of axisymmetric, stationary, force-free magnetospheres, the above equation implies that if H_φ is to be a poloidal function of θ alone, then the terms proportional to the inverse powers of r must vanish identically for choices of f and Ω.

General solutions to order $1/r^2$ can be considered by ignoring the $1/r^3$ term in the right hand side of Eq. (39). Here we require that Ω and f satisfy the relation

$$a^2\Omega\sin^2\theta\frac{d}{d\theta}(f\Omega\sin\theta) = \frac{d}{d\theta}(\frac{f}{\sin\theta}). \tag{40}$$

To solve this, let $g \equiv f\Omega\sin\theta$ and $h \equiv (\Omega\sin^2\theta)^{-1}$. With these definitions, Eq. (40) becomes

$$\frac{a^2}{h}\frac{d}{d\theta}g = \frac{d}{d\theta}(gh). \tag{41}$$

Integrating the above equation results in the relation

$$g = \frac{C_1\Omega\sin^2\theta}{\sqrt{\mid a^2-h^2\mid}}. \tag{42}$$

Consequently, for an arbitrarily chosen Ω,

$$f = \frac{C_1\sin\theta}{\sqrt{\mid (a\Omega\sin^2\theta)^2-1\mid}} \tag{43}$$

would make the $1/r^2$ term in Eq. (39) vanish. We can therefore successfully obtain the following function for H_φ. Explicitly,

$$\begin{aligned}
\frac{dH_\varphi^2}{d\theta} = 2f\Omega\sin\theta\frac{d}{d\theta}(f\Omega\sin\theta) = \frac{d}{d\theta}(f\Omega\sin\theta)^2 \Rightarrow \\
H_\varphi^2 = \pm H_0^2 + (f\Omega\sin\theta)^2.
\end{aligned} \tag{44}$$

The choice of Ω is determined by the Znajek regularity condition applied at the event horizon ($r_+ = M + \sqrt{M^2-a^2}$) so as to make B_φ finite in the well-behaved (even near the event horizon) Kerr-Schild coordinate system (see Znajek (1977); Komissarov (2004)). The Znajek condition can be written as

$$H_\varphi = \frac{\sin^2\theta}{\alpha_+}(2r_+M\Omega - a)B^r = \frac{\sin\theta}{\rho_+^2}(2r_+M\Omega-a)f, \tag{45}$$

where the subscript + indicates that the relevant quantities are to be evaluated at the event horizon and $\rho_+^2 = r_+^2 + a^2\cos^2\theta$. From Eqs. (44) and (45), we see that

$$\pm H_0^2 = \frac{\sin^2\theta}{\rho_+^4}[(4r_+^2M^2-\rho_+^4)\Omega^2-4r_+Ma\Omega+a^2]f^2. \tag{46}$$

We shall consider the solution for H_φ such that $H_0^2 = 0$ (it is easily seen that when $H_0 \neq 0$, the resulting solution does not permit a finite rate of energy extraction, and this type of situation will be dealt with in Subsection 6.2). This is possible if and only if the quantity in the square brackets in the above equation vanishes identically. Solving the resulting quadratic equation for Ω, we find two solutions, namely

$$\Omega_+ = \frac{a}{2Mr_+ + \rho_+^2}, \tag{47}$$

and

$$\Omega_- = \frac{a}{2Mr_+ - \rho_+^2} = \frac{1}{a\sin^2\theta}. \tag{48}$$

From Eq.(43) and the definition of f, we see that the only non-vanishing poloidal component of the magnetic field is given by

$$B_\pm^r = \frac{1}{\sqrt{\gamma}}f = \frac{1}{\sqrt{\gamma}}\frac{B_0\sin\theta(2Mr_+ \pm \rho_+^2)}{\sqrt{\mid (a\sin\theta)^4 - (2Mr_+ \pm \rho_+^2)^2\mid}}, \tag{49}$$

where we have relabeled C_1 as B_0. It is clear that Ω_- is an unphysical solution since B^r as given above is undefined everywhere.

6.1. *The Ω_+ Solution*

In this case, the non-vanishing components of the fields are

$$\begin{aligned}
B^r &= \frac{1}{\sqrt{\gamma}}\frac{B_0\sin\theta}{2\rho_+}\frac{\sqrt{a\Omega_H}}{\Omega_+} \\
E_\theta &= -\sqrt{\gamma}\Omega_+B^r \\
\alpha B_\varphi &= H_\varphi = -\sqrt{\gamma}\Omega_+B^r\sin\theta,
\end{aligned} \tag{50}$$

where $\Omega_H = a/2Mr_+$ is the angular velocity of the event horizon. When $a \ll M$

$$\begin{aligned}
B_+^r &\rightarrow \frac{1}{\sqrt{\gamma}}B_0\sin\theta\,,\ \text{and} \\
\Omega_+ &\rightarrow \frac{a}{8M^2}.
\end{aligned} \tag{51}$$

This is precisely the Blandford & Znajek (1977) monopole solution (Komissarov 2004). Therefore, the

solutions for the fields and currents corresponding to $\Omega = \Omega_+$ generalizes the Blandford-Znajek monopole solution to accommodate the case of a black hole for all values of $a^2 < M^2$.

A parallel approach to the study of the force-free magnetosphere has been developed via the Grad-Shafranov equation (see, e.g., Eq. (6.4) of MacDonald & Thorne (1982)). In our notation, the Grad-Shafranov equation takes the form (Uzdensky 2005)

$$\nabla \cdot [\frac{\alpha \nabla \psi}{\gamma_{\varphi\varphi}}(1 - \frac{(\Omega_+ + \beta^{\varphi})^2 \gamma_{\varphi\varphi}}{\alpha^2})] + \frac{(\Omega_+ + \beta^{\varphi})}{\alpha}\frac{d\Omega_+}{d\psi}(\nabla\psi)^2 + \frac{I}{\alpha\gamma_{\varphi\varphi}}\frac{dI}{d\psi} = 0. \quad (52)$$

Here, $I = H_\varphi$ and $\psi = A_\varphi$. By straightforward substitution and evaluation of the various terms in eq. (52), it is not difficult to see that our solution satisfies the Grad-Shafranov equation to order $1/r^2$.

From Eqs. (50) and (37), the angular dependence of energy extraction can be calculated. In the limit $r \gg M$, the result is

$$\frac{d^2\mathcal{E}}{dAdt} \approx \frac{a\Omega_H}{r^2}(\frac{B_0}{2})^2 \frac{\sin^2\theta}{\rho_+^2}. \quad (53)$$

From the above equation, it is clear that most of the energy extraction happens along the equatorial plane. The total rate of energy extraction can be obtained by integrating the above result, giving

$$\frac{d\mathcal{E}}{dt} = \frac{\pi B_0^2}{a r_+}[\arctan\frac{a}{r_+} - \frac{a}{2M}]. \quad (54)$$

In similar fashion, we see by integrating Eq. (38) that

$$\frac{d\mathcal{L}}{dt} = \frac{2\pi}{3}B_0^2\Omega_H + \frac{1}{\Omega_H}\frac{d\mathcal{E}}{dt}. \quad (55)$$

As a result of energy and angular momentum extraction from the black hole, the mass and the total angular momentum ($J = aM$) of the black hole changes by the amount

$$\frac{\delta M}{\delta t} = -\frac{d\mathcal{E}}{dt}, \text{ and } \frac{\delta J}{\delta t} = -\frac{d\mathcal{L}}{dt}, \quad (56)$$

respectively. From Eq. (55) and the above definitions, it clear that

$$\frac{\delta J}{\delta t} + \frac{2\pi}{3}B_0^2\Omega_H = \frac{1}{\Omega_H}\frac{\delta M}{\delta t}. \quad (57)$$

Therefore we get the familiar inequality (Christodoulou 1970)

$$\frac{\delta J}{\delta t} \leq \frac{1}{\Omega_H}\frac{\delta M}{\delta t}, \quad (58)$$

which ensures that the irreducible mass of the black hole is non-decreasing if the black hole evolves along a Kerr sequence in a reversible way. This process therefore cannot lead to the formation of a naked singularity.

6.2. *A Jet-Type Solution*

It is easily seen that $\Omega = \Omega_-$ removes all the r-dependence in the right-hand side of Eq. (39) to order $1/r^3$. As shown by Eqs. (48) and (49), $\Omega = \Omega_-$ is not a physical solution for the condition $H_0 = 0$. We now let $H_0 \neq 0$, and impose the Znajek condition, Eq. (45), for this case. Because our solutions involve both Ω_+ and Ω_-, the results continue to be valid only to order r^{-2}.

From Eq.(46) we see that

$$f^2 = \frac{\pm H_0^2 \rho_+^4}{a}\frac{\Omega_+\Omega_-^2}{(\Omega - \Omega_+)(\Omega - \Omega_-)}. \quad (59)$$

Here the $\pm$ factor is to ensure that $f^2 \geq 0$. Similarly we find from Eq. (43) that

$$f^2 = \frac{B_0^2}{a^2\sin^2\theta \mid (\Omega - \Omega_-)(\Omega + \Omega_-) \mid}. \quad (60)$$

Equating the right-hand sides of the last two equations, we see that

$$B_0^2 \mid \Omega - \Omega_+ \mid = \frac{\pm H_0^2\rho_+^4}{\sin^2\theta(2Mr_+ + \rho_+^2)} \mid \Omega + \Omega_- \mid . \quad (61)$$

It is important to remember that any Ω satisfying the above equation is consistent with Eq. (39) (to order $1/r^2$) and with Eq. (45). The above equation has the unique solution

$$\Omega_p = \frac{\tilde{A}\Omega_+ + \tilde{B}\Omega_-}{\tilde{A} - \tilde{B}}, \quad (62)$$

where

$$\frac{\tilde{A}}{B_0^2} = \begin{cases} +1, & \text{if } \Omega_p - \Omega_+ \geq 0 \\ -1, & \text{otherwise} \end{cases}, \text{ and}$$
$$\frac{\tilde{B}}{H_0^2\rho_+^4\Omega_+\Omega_-} = \begin{cases} +1, & \text{if } \Omega_p + \Omega_- \geq 0 \\ -1, & \text{otherwise.} \end{cases} \quad (63)$$

All other poloidal fields quantities are now uniquely determined by noting that f is given by Eq. (60). It is important to see if we can indeed satisfy the above conditions. A quick calculation shows that

$$\Omega_p - \Omega_+ = \frac{\tilde{B}(\Omega_+ + \Omega_-)}{\tilde{A} - \tilde{B}}, \text{ and}$$
$$\Omega_p + \Omega_- = \frac{\tilde{A}(\Omega_+ + \Omega_-)}{\tilde{A} - \tilde{B}}. \quad (64)$$

Therefore the choice $\tilde{A} = -B_0^2$ and $\tilde{B} = +H_0^2\rho_+^4\Omega_+\Omega_-$ is a valid one. We shall pick this choice for the remainder of the paper. Consequently, we have

$$\Omega_p = -\Omega_+\Omega_-\frac{[H_0^2\rho_+^4 - B_0^2a^2\sin^4\theta]}{[H_0^2\rho_+^4\Omega_+ + B_0^2 a\sin^2\theta]}. \quad (65)$$

Note that as $\theta \to 0$ and π, $\Omega_p \to -\Omega_- \to -\infty$. The form of f is determined by Eq. (60), and upon substitution of the explicit form of Ω ($= \Omega_p$), we find that in the limit as $\theta \to 0$ and π, $f \to \pm H_0\sqrt{a/\Omega_H}/2$.

The expression for the rate of total energy extraction is given by

$$\frac{d^2\mathcal{E}}{dAdt} = -H_\varphi\Omega_p\frac{1}{\sqrt{\gamma}}f \approx \frac{-1}{r^2}f^2\Omega_p\frac{(2Mr_+\Omega_p - a)}{\rho_+^2}. \quad (66)$$

As $\theta \to 0$ and π,

$$\frac{d^2\mathcal{E}}{dAdt} \to \frac{-1}{r^2}\frac{H_0^2 a}{4\Omega_H}\Omega_-(\Omega_- + \Omega_H) \quad (67)$$

A solution of this type has the following features: Energy extraction is less than zero near the poles, i.e., energy is being fed into the system, indicating a reverse jet type situation. Also, the total rate of energy and angular momentum “insertion” is not calculable since the above integral is divergent along the poles. This solution is therefore unphysical.

7. DISCUSSION

Based on the 3+1 equations as written by Komissarov (2004), we have rederived the constraint equation relating the toroidal magnetic field to the charge and current densities in a force-free magnetosphere around a spinning black hole. Known solutions to the constraint equation for the force-free magnetosphere include the monopole and the parabolic solutions obtained in the orginal paper by Blandford & Znajek (1977), and the solution by Beskin et al. (1992) for a black hole surrounded by a magnetized, conducting accretion disk. We have discovered a solution to the constraint equation that generalizes the "monopole solution" originally derived by Blandford & Znajek (1977). This solution satisifes the Znajek (1977) regularity condition at the event horizon, even in the limit $a/M \ll 1$ (contrary to the statement of Blandford and Znajek).

Komissarov (2001) has used a time-dependent numerical simulation to calculate the electromagnetic extraction of energy for a monopole magnetic field at different values of a/M. Our value of Ω_+/Ω_H at $\theta = 0.5$ ranges from 0.5 to 0.58 when a/M varies from 0.1 to 0.9, in comparison with the numerical value of 0.52 for $a/M = 0.9$ at $r = 10$. The value of H_φ for our Ω_+ solution is $\approx 25\%$ larger than the numerical value of Komissarov (2001) when $a/M = 0.9$. These discrepancies, though not large, may reflect the finite value of $r = 10$ used in Komissarov's work, whereas our solution holds in the asymptotic limit of large r.

For the Ω_+ solution, energy and angular momentum is extracted preferentially along the equatorial directions of the spinning black hole. As such, it does not account for galactic black holes and active galactic nuclei that display radio jets. Time-dependent numerical solutions employing accretion of magnetized plasma into the ergosphere seem to indicate the presence of such jet-like features (Semenov et al. 2004). In future work, analytic solutions that exhibit jet-like structures will be studied using the techniques developed in this paper.

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